Business Application of Maxima and Minima

13

SLOB Mapped against the Module

To equip oneself with application-oriented knowledge of Differential Calculus to facilitate management decisions for optimisation through resource allocation, managing competition, work scheduling and managing cost overrun, demand estimation, production and cost analysis etc.

Module Learning Objectives

After studying this module, the students will be able to:

- Understand the areas of application of Maxima and Minima with respect to business.
- Solve the problems of optimization of Functions of One Variable with no constraint.
- Solve the problems of optimization of Multivariate Functions with no constraint.
- Solve the problems of optimization of Multivariate Functions subject to equality constraint.

Business Application of Maxima and Minima

13

From the legal point of view the term Business is defined as "Any activity or enterprise entered into for Profit". So whenever we talk about business application of something, it boils down to profit making by an enterprise through its activities. Actually "Enterprise" is the term in the commercial world to describe a project or venture undertaken for gain. This is synonymous with the term Firm.

Thus business application of *Maxima and Minima* mainly refers to the process of finding the optimal strategies by a Firm to achieve its goal or objective. Economists believe that Business Firms always attempt to maximize their profit (or minimize their losses) They however do realise that businessmen have other goals like maximization of sales, maximization of firms' growth rate etc. for their survival which subsequently provide a quiet life. Yet the traditional economic theory assumes profit maximization as the sole objective of the business firms. Such an assumption turns out to be very useful and convenient due to the following reasons –

- In actual practice firms come fairly close to act like profit maximizers.
- It is helpful in correctly predicting the behaviour of the business firms with regard to the quantity of output produced and prices charged in the real world.

No wonder therefore, the whole traditional economic theory is developed on the basis of profit maximization (or loss minimization) hypothesis. As a result mathematical techniques are observed to be used to a great extent. The first condition for such techniques to be useful is that economic relationships or functions are expressed in algebraic form which in turn demands the objectives are well defined and transformed into quantitative statements. There can be two different situations under which the optimization of objectives are carried out. These are given as – (1) Unconstrained Optimization and (2) Constrained Optimization

Unconstrained Optimization can further have sub divisions as given below -

(A) Optimization of single variable objective function

(B) Optimization of objective functions having multiple variables.

Similarly Constrained Optimization has the following sub divisions.

(A) Equality constrained Optimization

(B) Inequality constrained Optimization

(C) Static Optimization

(D) Dynamic Optimization

Of all these types of optimization situations, concept of Maxima and Minima of differential calculus is used for both types of Unconstrained Optimization. Use of differential calculus is observed for Constrained Optimization with Equality Constraint. Though the same technique can be used for Constrained Optimization with Inequality Constraints, but in practice that has not gained much popularity.

As mentioned above, differential calculus is used a lot for study of the optimization in Economics. So recapitulation of it is necessitated. The standard formulae and rules are given below.

Review of Standard formulae and rules of Differentiation

1. If $y = f(x) = x^n$ then $dy/dx = f'(x) = nx^{n-1}$
2. If $y = f(x) = e^x$ then $dy/dx = f'(x) = e^x$, where e = constant = 2.718 (approx.)
3. If $y = f(x) = e^{mx}$ then $dy/dx = f'(x) = me^{mx}$, where 'm' is a constant
4. If $y = f(x) = \log_e x$ or ln x then $dy/dx = f'(x) = 1/x$
5. If $y = f(x) = k$ then $dy/dx = f'(x) = 0$, where k is a constant
6. If $y = f(x) = p(x) \pm q(x)$ then $dy/dx = f'(x) = p'(x) \pm q'(x)$
7. In case of Partial differentiation, all the above formulae / rules hold good with the exception that all the terms present in the given function are considered as constant except the ones with the participating variable of partial differentiation. Some examples are -
   (a) If $U = xy$ then $\partial U/\partial x = y$ and $\partial U/\partial y = x$
   (b) If $U = x^a y^b$ then $\partial U/\partial x = ax^{a-1}y^b$ and $\partial U/\partial y = bx^a y^{b-1}$ [where 'a' and 'b' are constants]
   (c) If $U = px + qy$ then $\partial U/\partial x = p$ and $\partial U/\partial y = q$ [where 'p' and 'q' are constant]
   (d) If $U = a\sqrt{x} + b\sqrt{y}$ then $\partial U/\partial x = a/2.(x)^{-1/2}$ and $\partial U/\partial y = b/2.(y)^{-1/2}$ [where 'a' and 'b' are constant].

Unconstrained Optimization

A major part of economic analysis assumes not only maximizing behaviour on the part of the economic actors but also unconstrained optimization or mathematical optimization. Such type of optimization is also known as Unbounded Maxima technique. As mentioned above, there can be two different situations involving either single variable or multiple variables.

1. Optimization of Functions involving Single Independent Variable

When the objective function is given as an Algebraic Function and no constraints are imposed then Calculus approach of derivatives is used to optimize the function. The conditions of optimization are two fold and given as

1. **Necessary condition:** The 1st Order Derivative should be Zero, that is $\frac{dy}{dx} = 0$
   This is applicable for both the situations of Maximization as well as Minimization.
2. **Sufficient condition:** $\frac{d^2y}{dx^2}$ **< 0** for ***Maxima*** and $\frac{d^2y}{dx^2}$ **> 0** for ***Minima*** at the value of x obtained from (1)

The value/s of x obtained from the Necessary Condition (1) written above, is/are called Critical Value/s. The sign of 2nd Order Derivative is checked by putting the Critical value/s in it and subsequently decision regarding Maxima or Minima is taken as per Sufficient Condition (2) above.

It can be mentioned that when $\frac{d^2y}{dx^2} = 0$ then there exist neither a Maxima nor a Minima. Such a point is known as **Point of Inflexion**.

Illustration 1 (Maximization problem)

The demand (rides per day) of Roller Coaster Ride in an Entertainment Park in one of the metro cities is given by the equation $q = -450p + 41500$, where p = Price per ride in ₹ What price should have been charged to maximize the Total Revenue?

Solution:

Total Revenue is algebraically expressed as a function of price as follows

R(p) = Price per ride × Demand Or, $R(p) = p \times q$ Or, $R(p) = p(-450p + 41500)$ Or, $R(p) = 41500p - 450p^2$

Differentiating both sides with respect to 'p' we get

$$\frac{d}{dp}[R(p)] = 41500 - 900p \text{...} (i)$$

As per the necessary condition of optimization, $\frac{d}{dp}[R(p)] = 0$ Or, $41500 - 900p = 0$ Or, $p = 46.11$

To ascertain whether the value of p obtained corresponds to a maxima, we have to take help of sufficient condition written above.

Again differentiating both sides of (i) with respect to 'p' we get, $\frac{d^2}{dp^2}[R(p)] = -900 < 0$

So there exist a Maxima at $p = 46.11$

Thus the price to be charged to maximize the Total Revenue is ₹46.11/-

Illustration 2 (Minimization problem)

Assume the Cost (₹) of manufacturing x numbers of a product per day is $C(x) = 14400 + 550x + 0.01x^2$**.** How many of the product should be manufactured per day so that the Average Cost is minimum? Also find the values of the Average Cost and the Total Cost at this level of production.

Solution:

Cost function is given to be $C(x) = 14400 + 550x + 0.01x^2$

So Average Cost function = C(x) / x Or, $AC(x) = (14400 + 550x + 0.01x^2) / x$ Or, $AC(x) = 14400/x + 550 + 0.01x$

This is the Objective function which has to be minimized.

Differentiating both sides of the above function with respect to 'x' we get

$$\frac{d}{dx}[AC(x)] = -14400/x^2 + 0.01 \text{ -------- } (i)$$

As per the necessary condition of optimization, $\frac{d}{dx}[AC(x)] = 0$ Or, $-14400/x^2 + 0.01$ Or, $0.01x^2 = 14400$

Or, $x^2 = 14400/0.01$ Or, $x = \pm\sqrt{1440000}$ Or, $x = \pm 1200$

But x being the quantity cannot be negative. Hence x = 1200

To ascertain whether this value of x corresponds to minima, we have to take help of the sufficient condition mentioned above.

Again differentiating both sides of (i) with respect to 'x' we get, $\frac{d^2}{dx^2}[AC(x)] = 28800/x^3$

For x = 1200, the value of 2nd order Derivative is $\frac{d^2}{dx^2}[AC(1200)] = 28800/(1200)^3 = 1.67 \times 10^{-5} > 0$

So there exist a Minima to the Objective Function at x = 1200

Hence 1200 units should be produced per day to minimize the Average Cost.

At this level of production, Average Cost = $[AC(X)]_{at\ x = 1200}$ = 14400/1200 + 550 + 0.01 × 1200 = ₹574 per unit

Also at this level of production, Total Cost = $[C(x)]_{at\ x = 1200}$ = 14400 + 550 × 1200 + 0.01 × 1200^2 = ₹6,88,800/-

2. Optimization of Functions involving Multiple Independent Variables

For the situations where the Objective Function involves more than one Independent variable (say two) and the change in the dependent variable is the joint impact of changes in both the variables then the approach towards optimization of the Objective Function, though takes help of differential calculus, but is not exactly same as that used for the case of single Independent variable.

In fact the measurement of the independent impact of one variable is not possible without assuming that the other variable remains unchanged. As example we consider the case of Sales of a commodity which is a multivariate function of Price and Advertising Expenditure. Now the impact of change in Price over Sales cannot be measured if we do not assume the Advertising Expenditure to remain same. Partial Derivative of a function explains the same logic mathematically.

The conditions of optimization in this case are given as follows –

1. The values of the 1st Order Partial Derivatives should be Zero. That is $\frac{\partial f}{\partial x}$ or $f_x = 0$ and $\frac{\partial f}{\partial y}$ or $f_y = 0$, when the Objective Function is Bivariate. If the number of variables are more, the Partial Derivatives of the function with respect to those variables should also be zero.

   From the above equations the Critical Values of x and y are determined. In other words the coordinates of the Critical Point (a,b) are determined, where a = Value of x and b = Value of y for a Bivariate function.

2. Next find out all possible 2nd Order Partial Derivatives. For a Bivariate Function these are

   $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}, $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}, $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy} and $\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx}. Calculate the numerical values of these by putting x = a and y = b

3. Assume:- $A = f_{xx}(a,b)$, $B = f_{xy}(a,b)$ and $C = f_{yy}(a,b)$ and find the value of $D = AC – B^2$
   - If **D > 0** and **A, C > 0** then there is a **local Minima at (a,b)**
   - If **D > 0** and **A, C < 0** then there is a **local Maxima at (a,b)**
   - If **D < 0** then **(a, b) is a Saddle Point**
   - If **D = 0** then **the test fails**.

[**Note:** The expression $D = AC – B^2$ can also be represented in the form of a determinant as follows –

$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix}$ Or, $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$. As f_{xy} is equal to f_{yx} always, we represent both as B. This determinant actually corresponds to a (2 × 2) Matrix called Hessian Matrix. So Hessian Matrix of order 2 is given as $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ and the same is applicable when there are two variables in the problem. For problems with three variables, we will come across Hessian Matrix of order 3.]

Illustration 3 (Maximization problem)

A company produces two products x and y. The total Profit (in ₹ '000) earned by the company is expressed algebraically by the function $P = 100x - x^2 - 2xy + 200y - 3y^2$. Find the Profit maximizing quantities of the products. Also find out the maximum Profit.

Solution:

Profit function is given as :- $P = 100x - x^2 - 2xy + 200y - 3y^2$

Differentiating the function partially with respect to x we get,

$P_x = 100 - 2x - 2y$ ------- (I)

Also differentiating the function partially with respect to y we get

$P_y = - 2x + 200 - 6y$ ----- (II)

To determine the Critical Point we have $P_x = 0$ and $P_y = 0$

So, $100 - 2x - 2y = 0$ Or. $x + y = 50$ --------- (1) and $- 2x + 200 - 6y = 0$ Or, $x + 3y = 100$ ---------- (2)

(2) – (1) gives, $2y = 50$ Or, $y = 25$

Putting $y = 25$ in (1) we get $x = 25$

Thus Critical Point is (25, 25)

To check whether this point is a local Maxima, we have to find out the values of the 2nd Order Partial Derivatives at this point.

Again differentiating (I) partially with respect to x we get $P_{xx} = -2$ Or, $A = -2$ (Let) Or, $A < 0$

Similarly differentiating (II) partially with respect to y we get $P_{yy} = -6$ Or, $C = -6$ (Let) Or, $C < 0$

Also differentiating (I) partially with respect to y we get $P_{xy} = -2$ Or, $B = -2$ (Let)

So $D = AC - B^2 = (-2) \times (-6) - (-2)^2 = 8 > 0$

Hence $D > 0$ and $A, C < 0$

Thus there is a local Maxima at the already determined Critical Point (25, 25).

Required Profit maximizing quantities of the products are x = 25 units and y = 25 units.

Also, Maximum Profit = Value of the function P at x = 25 & y = 25 = $100 \times 25 - 25^2 - 2 \times 25 \times 25 + 200 \times 25 - 3 \times 25^2$

= ₹ 3750 (₹000)

Illustration 4

Find the Critical Points for the function $f(x,y) = x^2y - 2xy^2 + 3xy + 4$. Examine the presence of Local Maxima and Minima among the Critical Points.

Solution:

For the Critical Points of $f(x,y) = x^2y - 2xy^2 + 3xy + 4$, at first we have to find out the1st order Partial Derivatives.

Partial differentiation of the given function f(x,y) with respect to x gives $f_x = 2xy - 2y^2 + 3y$ --------- (1)

Partial differentiation of the given function f(x,y) with respect to y gives $f_y = x^2 - 4xy + 3x$ --------- (2)

As per the 1st Order Partial Derivative rule we have,

$f_x = 0$ Or, $2xy - 2y^2 + 3y = 0$ Or, $y(2x - 2y + 3) = 0$ --- (3) & $f_y = 0$ Or, $x^2 - 4xy + 3x = 0$ Or, $x(x - 4y + 3) = 0$ --- (4)

From (3) & (4) we get $x = 0$, $y = 0$ and $2x - 2y + 3 = 0$ ----- (5) as well as $x - 4y + 3 = 0$ ------ (6)

(5) – 2 × (6) gives, $y = 1/2$ and $x = -1$

Again putting $x = 0$ in (5) we get $y = 3/2$ and putting $y = 0$ in (6) we get $x = -3$

So the **Critical Points** of the function are **(0,0), (- 3, 0), (0, 3/2) and (-1, 1/2)**

Now the 2nd Order Partial Derivatives are found out by differentiating (1) and (2) partially as follows –

Partially differentiating (1) w.r.t 'x' gives $f_{xx} = 2y$ and partially differentiating w.r.t 'y' gives $f_{xy} = 2x - 4y + 3$

Partially differentiating (2) w.r.t 'y' gives $f_{yy} = -4x$

Let, $A = f_{xx}$, $B = f_{xy}$ and $C = f_{yy}$.

For the Critical Point (0,0) we have $A = (2y) = 2.0 = 0$, $B = (2x - 4y + 3) = 2.0 - 4.0 + 3 = 3$ & $C = (-4x) = -4.0 = 0$

$D = AC - B^2 = 0.0 - 3^2 = -9 < 0$ So there is a **Saddle Point** at (0,0)

For the Critical Point (-3, 0) we have $A = 0$, $B = 2.(-3) - 4.0 + 3 = -3$ & $C = -4.(-3) = 12$

$D = AC - B^2 = 0.(12) - (-3)^2 = -9 < 0$ So there is a **Saddle Point** at (- 3, 0)

For the Critical Point (0, 3/2) we have $A = 2.\ 3/2 = 3 > 0$, $B = 2.0 - 4.3/2 + 3 = -3$ & $C = 0$

$D = AC - B^2 = 3.0 - (-3)^2 = -9 < 0$ So there is a **Saddle Point** at (0, 3/2)

For the Critical Point (-1, 1/2) we have $A = 2.1/2 = 1 > 0$, $B = 2.(-1) - 4.1/2 + 3 = -1$ & $C = -4.(-1) = 4 > 0$

$D = AC - B^2 = 1.4 - (-1)^2 = 3 > 0$ Thus $D > 0$ as well as A & C > 0. So there is a local **Minima** at (-1, 1/2).

Equilibrium of a Firm

Equilibrium means a "State of Rest". In this State the forces working in opposite directions are exactly in balance so that there is no tendency to move in any direction. A firm is said to be in equilibrium when it selects a particular level of output at which it would like to "Stay" or "Rest". There is no incentive for the firm to increase or decrease output from that level. In other words, **A firm is in equilibrium when, given the demand and cost conditions, it produces that level of output at which the Profit is maximised**.

From some other point of view, a firm is supposed to be in equilibrium when its objective is optimised. The objective of a firm may be many, but the Neo Classical theory of Economics assumes maximization of Profit is the sole objective of it. The level of output and the price charged corresponding to the equilibrium are called the Equilibrium Output and the Equilibrium Price respectively.

Some important terminology related to the concept of Profit maximisation are as follows.

Revenue*:* The Revenue is defined as the money earned by selling certain quantity of output. More precisely it should be called the "Sales Revenue" and it must not be mixed up with other similar concepts of earning money like income, profit etc.

Revenue involves three inter-related concepts – Total Revenue (TR), Average Revenue (AR) & Marginal Revenue (MR).

Total Revenue is the product of Price (P) and the Quantity Sold (x). Thus TR = P.x

Average Revenue is the Revenue earned per unit sale. So AR = TR / x Or, AR = Px /x Or, AR = P = Price

Hence Average Revenue can be termed as the Price or vice versa. Thus Average Revenue curve is the same as the Demand curve.

Marginal Revenue is defined as the revenue earned by selling an additional unit of output. In other words, it represents the rate of change of Total Revenue with respect to Output which is nothing but the derivative of Total Revenue with respect to Output. Hence MR = $\frac{d}{dx}$ (TR)

Mathematically Revenue is expressed as a function of Output. That is R = f(x)

Cost of Production: Like Revenue Cost, too has three basic concepts – Total Cost (TC), Average Cost (AC) and Marginal Cost (MC)

Total Cost has two components: Fixed and Variable. So TC = TFC + TVC

Average Cost is given as AC = TC/ x Or, AC = TFC/ x + TVC/ x Or, AC = AFC + AVC

Marginal Cost is the cost of producing an additional unit of output.

Mathematically, MC = $\frac{d}{dx}$ (TC) = $\frac{d}{dx}$ (TFC+TVC)

Cost of Production is expressed as a function of Output in order to know its behaviour at different levels of output produced or capacity utilization. So C = f(x) is the mathematical way of representation of Cost.

Profit: It is the residual after deducting Cost from the Revenue. It is represented as π = f(x)

Total Profit = T π = TR – TC Or, T π = TR – TC Or, T π = x (AR) – x (AC) Or, T π = x (AR – AC)

Average Profit = A π = T π/x Or, A π = (TR – TC)/ x Or, A π = TR/x – TC/x Or, A π = AR – AC

Marginal Profit = M π = $\frac{d}{dx}$ (T π) Or, M π = $\frac{d}{dx}$ (TR – TC) Or, M π = $\frac{d}{dx}$ (TR) - $\frac{d}{dx}$ (TC) Or, M π = MR – MC

1. Condition for Firm's Equilibrium

It has already been discussed that a Firm is said to be in Equilibrium when it maximizes its Profit. As per the conditions of mathematical approach it requires that $\frac{d}{dx}$ (T π) = 0 and $\frac{d^2}{dx^2}$ (T π) < 0

Now, T π = TR – TC

Differentiating both sides with respect to x we get

$$\frac{d}{dx}(T\pi) = \frac{d}{dx}(TR) - \frac{d}{dx}(TC) \text{ Or, } \frac{d}{dx}(T\pi) = MR - MC$$

But, (T π) = 0 Or, MR – MC = 0 Or, **MR = MC** is the **Condition for a Firm's Equilibrium**

This is the **necessary condition** of a Firm's Equilibrium.

For the **sufficient condition** of the Firm's Equilibrium we should have $\frac{d^2}{dx^2}(T\,\pi) < 0$ Or, $\frac{d^2}{dx^2}(TR - TC) < 0$

$$\text{Or, } \frac{d^2}{dx^2}(TR) - \frac{d^2}{dx^2}(TC) < 0$$

$$\text{Or, } \frac{d^2}{dx^2}(TR) < \frac{d^2}{dx^2}(TC)$$

$$\text{Or, } \frac{d}{dx}\left[\frac{d(TR)}{dx}\right] < \frac{d}{dx}\left[\frac{d(TC)}{dx}\right]$$

$$\text{Or, } \frac{d}{dx}[MR] < \frac{d}{dx}[MC]$$

The conditions written above are meant for single variable type situations of optimization. In case the number of variables is more than one then the Condition of Firm's Equilibrium for the following situations are –

1. **For Multi-plant Monopolist Firm producing the same product in all the Plants**

The **necessary condition** is $\mathbf{MR = MC_1 = MC_2}$, where MC_1 and MC_2 = Respective Marginal Costs of Production in Plants 1 & 2 (for the sake of simplicity the discussion has been confined to two Plants. When there are more than two Plants, then also the condition is applicable)

The **sufficient condition** is given as $\frac{d^2}{dx^2}\mathbf{(TR)} < \frac{d^2}{dx_1^2}\mathbf{(TC_1)}$ and $\frac{d^2}{dx^2}\mathbf{(TR)} < \frac{d^2}{dx_2^2}\mathbf{(TC_2)}$ [x_1 & x_2 are the quantities produced in the Plants 1 and 2 respectively. Also $x = x_1 + x_2$]

2. **For Price discriminating Monopolist Firm selling the same product at different prices in different markets**

The **necessary condition** is $\mathbf{MC = MR_1 = MR_2}$, where MR_1 and MR_2 are the respective Marginal Revenues for the two different market segments having Demand Functions given as p_1 and p_2

The **sufficient condition** is given as $\frac{d^2}{dx_1^2}\mathbf{(TR_1)} < \frac{d^2}{dx^2}\mathbf{(TC)}$ and $\frac{d^2}{dx_2^2}\mathbf{(TR_2)} < \frac{d^2}{dx^2}\mathbf{(TC)}$ [x_1 & x_2 are the quantities sold at the prices p_1 and p_2 respectively. Also $x = x_1 + x_2$]

[**N.B –** Monopolist Firms operate in a Monopoly market structure in which there is a single seller, there are no close substitutes for the commodity it produces and there are barriers to entry.]

Illustration 5

A firm has the Cost function $C = x^3/3 - 7x^2 + 111x + 50$ and Demand function $x = 100 - p$. Determine the Equilibrium Output, Price and Profit earned.

Solution:

Demand function is $x = 100 - p$ Or, $p = 100 - x$

So, Total Revenue = TR = p.x Or, TR = $(100 - x)x$ Or, TR = $100x - x^2$

Also Profit = Total Revenue – Cost Or, $\pi = TR - C$ Or, $\pi = (100x - x^2) - (x^3/3 - 7x^2 + 111x + 50)$

Or, $\pi = - x^3/3 + 6x^2 - 11x - 50$

Differentiating both sides with respect to x we have $\frac{d}{dx}(\pi) = - x^2 + 12x - 11$ -------- (1)

As per the necessary condition of maximization we have $\frac{d}{dx}(\pi) = 0$ Or, $- x^2 + 12x - 11 = 0$ Or, $(x - 1)(x - 11) = 0$

So the critical values are x = 1 and x = 11

Now differentiating both sides of (1) we have $\frac{d^2}{dx^2}(\pi) = - 2x + 12$

When x = 1 then $\frac{d^2}{dx^2}(\pi) = - 2.1 + 12 = 10 > 0$

So by the sufficient condition of 2nd Order Derivative test there is a minima at x = 1

When x = 11 then $\frac{d^2}{dx^2}(\pi) = -2.11 + 12 = - 10 < 0$

So by the sufficient condition of 2nd Order Derivative test there is a maxima at x = 11

Thus Profit (π) is Maximum when x = 11 units. This is the required Equilibrium Output.

Equilibrium Price = $p_{Equilibrium} = [100 - x]_{at\ x = 11} = 100 - 11 =$ ₹89

Equilibrium Profit = $(\pi)_{Max.} = [- x^3/3 + 6x^2 - 11x - 50]_{at\ x = 11} = - (11)^3/3 + 6(11)^2 - 11.11 - 50 =$ ₹ 111.33

[**Note** – The equilibrium output can be determined by using the relation MR = MC. Subsequently this value of output can be substituted in the Demand and Profit functions to obtain Equilibrium Price and Profit.]

Illustration 6

A manufacturer produces a liquid commodity at two different Plants located in the two regions of the country. The selling price (in ₹/litre) of the product is given by the equation $p = 200 - 0.8x$, where $x = x_1 + x_2$ = Total production of the two Plants together. The Cost functions of the 2 Plants are given as $C_1 = 0.3(x_1)^2 + 60x_1 + 5000$ and $C_2 = 0.5(x_2)^2 + 30x_2 + 8000$. Determine the quantities produced by the two Plants which will put the manufacturer into an equilibrium condition.

Solution:

Total Revenue = TR = p.x Or, TR = $(200 - 0.8x)x$ Or, TR = $200x - 0.8x^2$

Marginal Revenue (MR) = d(TR)/dx = $\frac{d}{dx}(200x - 0.8x^2) = 200 - 1.6x$ -------- (1)

Marginal Cost for the 1st Plant (MC_1) = $dC_1/dx_1 = \frac{d}{dx_1}[0.3(x_1)^2 + 60x_1 + 5000] = 0.6x_1 + 60$ --------- (2)

Marginal Cost for the 2nd Plant (MC_2) = $dC_2/dx_2 = \frac{d}{dx_2}[0.5(x_2)^2 + 30x_2 + 8000] = 1.0x_2 + 30$ -------- (3)

As per the Necessary Condition of Equilibrium of a Multi Plant Firm we have $MR = MC_1 = MC_2$

So from (1) & (2) we get $200 - 1.6x = 0.6x_1 + 60$ Or, $200 - 1.6(x_1 + x_2) = 0.6x_1 + 60$ Or, $1.1x_1 + 0.8x_2 = 70$ --- (4)

Also from (2) & (3) we get, $0.6x_1 + 60 = 1.0x_2 + 30$ Or, $x_2 = 0.6x_1 + 30$ -------- (5)

Substituting $x_2 = 0.6x_1 + 30$ from (5) in (4) we get, $1.1x_1 + 0.8(0.6x_1 + 30) = 70$ Or, $x_1 = 29.1$

Substituting $x_1 = 29.1$ in (5) we get, $x_2 = 47.5$

Now $\frac{d^2}{dx^2}(TR) = \frac{d}{dx}(MR) = \frac{d}{dx}(200 - 1.6x) = -1.6$

Also $\frac{d^2}{dx_1^2}(TC_1) = \frac{d}{dx_1}(MC_1) = \frac{d}{dx_1}(0.6x_1 + 60) = 0.6$

Again $\frac{d^2}{dx_2^2}(TC_2) = \frac{d}{dx_2}(MC_2) = \frac{d}{dx_2}(1.0x_2 + 30) = 1.0$

Hence from above we get $\frac{d^2}{dx^2}(TR) < \frac{d^2}{dx_1^2}(TC_1)$ and $\frac{d^2}{dx^2}(TR) < \frac{d^2}{dx_2^2}(TC_2)$

Thus the sufficient condition for Equilibrium is satisfied.

Required equilibrium outputs are **29.1 litres** and **47.5 litres** from the Plants 1 and 2 respectively.

[**Note –** The problem can be solved by using the procedure explained in Illustration 3]

Illustration 7

A discriminating Monopolist is able to separate its customers into two markets with respective Demand functions as $x_1 = 16 - 0.2p_1$ and $x_2 = 9 - 0.05p_2$. Total Cost function is $TC = 20 + 20x$, where $x = x_1 + x_2$. Determine the Equilibrium Price of the product in the two markets. Also determine the Equilibrium Profit.

Solution:

Demand function of the 1st market is $x_1 = 16 - 0.2p_1$ Or, $p_1 = 80 - 5x_1$ ....(1)

Also demand function of the 2nd market is $x_2 = 9 - 0.05p_2$ Or, $p_2 = 180 - 20x_2$ ....(2)

From the 1st market, Revenue earned = $TR_1 = p_1x_1 = (80 - 5x_1).x_1 = 80x_1 - 5(x_1)^2$ ....(3)

From the 2nd market, Revenue earned = $TR_2 = p_2x_2 = (180 - 20x_2).x_2 = 180x_2 - 20(x_2)^2$ ....(4)

So Marginal Revenue for the 1st case = $MR_1 = \frac{d}{dx_1}(TR_1) = \frac{d}{dx_1}[80x_1 - 5(x_1)^2] = 80 - 10x_1$ ....(5)

Also Marginal Revenue for the 2nd case = $MR_2 = \frac{d}{dx_2}(TR_2) = \frac{d}{dx_2}[180x_2 - 20(x_2)^2] = 180 - 40x_2$ ....(6)

Total Cost Function is $TC = 20 + 20x$

So Marginal Cost (MC) = $\frac{d}{dx}(TC) = \frac{d}{dx}(20 + 20x) = 20$ ....(7)

As per the necessary condition of equilibrium of a Price Discriminating Monopolist Firm we have

$MC = MR_1 = MR_2$

From (5) and (7) we have, $80 - 10x_1 = 20$ Or, $x_1 = 6$

From (6) and (7) we have, $180 - 40x_2 = 20$ Or, $x_2 = 4$

$$\frac{d^2}{dx_1^2}(TR_1) = \frac{d}{dx_1}(MR_1) = \frac{d}{dx_1}(80 - 10x_1) = -10$$

$$\frac{d^2}{dx_2^2}(TR_2) = \frac{d}{dx_2}(MR_2) = \frac{d}{dx_2}(180 - 40x_2) = -40$$

$$\frac{d^2}{dx^2}(TC) = \frac{d}{dx}(MC) = \frac{d}{dx}(20 + 20x) = 20$$

So from above we get $\frac{d^2}{dx_1^2}(TR_1) < \frac{d^2}{dx^2}(TC)$ and $\frac{d^2}{dx_2^2}(TR_2) < \frac{d^2}{dx^2}(TC)$

Thus the sufficient condition for equilibrium is satisfied.

Hence the equilibrium price for the 1st market = $p_1 = (80 - 5x_1)_{At\ x1 = 6} = 80 - 5.6 =$ ₹ **50/-**

Also the equilibrium price for the 2nd market = $p_2 = (180 - 20x_2)_{At\ x2 = 4} = 180 - 20.4 =$ ₹ **100/-**

Now Total Revenue of the Firm = $TR = TR_1 + TR_2 = 80x_1 - 5(x_1)^2 + 180x_2 - 20(x_2)^2$

Also Total Cost = $TC = 20 + 20x = 20 + 20(x_1 + x_2)$

Thus Profit Function is given as $\pi = TR - TC$ Or, $\pi = [80x_1 - 5(x_1)^2 + 180x_2 - 20(x_2)^2] - [20 + 20(x_1 + x_2)]$

Equilibrium Profit = $(\pi)_{Equilibrium} = 80.6 - 5.6^2 + 180.4 - 20.4^2 - 20 - 20.(6 + 4) =$ ₹ 480

[**Note –** An alternative method of solving the problem is the procedure used in Illustration 3]

Constrained Optimization

The Optimization technique discussed earlier in this module is an example of Unbounded Maxima. It assumed that the Firm is capable of not only finding its Equilibrium Output but also producing it without any restrictions. All the resources required to produce the given level of output are at its command. It has no shortage of inputs, energy, labour, transport, liquidity etc. The economic actor under Neo Classical theory thus happened to be an unbounded maximizer.

In practical life the conditions are not exactly like the ones assumed in case of Unbounded Maxima. Firms need to compete for the procurement of inputs in the market. Scarcity of good quality energy is a very regular affair in our country. Availability of labours of desired skill is another questionable area. Transportation of goods from one place to the other, particularly from production centres to the warehouses, is troublesome. Credits of required amount may not be available at the time when it is actually required or the rate of interest may be exorbitantly high.

Thus working under restrictions is the most common phenomenon as far as operations of a Firm is concerned. A Firm has to work to its best possible ability even after facing various types of practical difficulties. Optimization with restrictions of different kind is known as *Constrained Optimization.*

Mathematically the constraints are expressed either in the form of Equations (which are also known as Equalities) or in the form of Inequalities. Problems of equality constraints are dealt with *Lagrangian Multipliers* and those involving inequality constraints are solved by Linear Programming techniques.

As different types of Linear Programming techniques are dealt separately in different modules of this study material, we will confine our discussion only to the Lagrangian Multipliers in this module.

1. Optimization with Single Equality Constraint

Some of the situations which come under the purview of such type of optimization can be –

- A consumer has to choose how much to buy of each product such that it satisfies the budget constraint. In other words it is a case of Maximization of Utility subject to budget constraint.
- A firm would look to minimize its cost of production subject to a given output level which can also be termed as Minimization of Cost subject to output constraint.
- A manufacturer would try to maximize its production with whatever quantity of a scarce material available with him/her. That means a problem of Maximization of Outputs subject to resource constraint.

In all the above situations Lagrangian Multiplier Method can be successfully used for optimization. To explain the steps involved, we take help of the following example.

A consumer has a choice of two commodities X & Y. Prices of these are p and q respectively. The person has limited money (M) and wants to procure maximum possible quantities of X & Y with the amount he / she has.

The problem can be rewritten as –

Maximize Utility function U = f(X,Y) subject to the constraint pX + qY = M

Steps to be followed by Lagrangian Multiplier Method are –

1. Transform the Constraint equation to a form with 0 on the R.H.S. In this case it is pX + qY – M = 0
2. Multiply L.H.S of the transformed Constraint Equation by Lagrange's Multiplier (λ). In this case it takes the form $\lambda.(pX + qY - M)$
3. From the original Objective Function subtract the one obtained in the previous step to form Lagrangian function given by $L(X, Y, \lambda) = f(X,Y) - \lambda.(pX + qY - M)$

   [It can be noted that L(X, Y, λ) = f(X,Y) when the constraint holds i.e pX + qY = M. Hence maximization of Lagrangian function ultimately maximises the original objective function f(X,Y)]
4. Find the critical values of the unknowns (X, Y and λ) using the 1st Order conditions i.e all the partial derivatives are equal to zero.

   Thus, $\partial L/\partial X = 0$ Or, $f_X - \lambda p = 0$

   $\partial L/\partial Y = 0$ Or, $f_Y - \lambda q = 0$

   and $\partial L/\partial \lambda = 0$ Or, $pX + qY - M = 0$

   Solving the above three equations one can get the values of three unknowns X, Y and λ.
5. Now find a Bordered Hessian Matrix (HB) given as –

$$HB = \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix}$$

   It can be noted that and are the partial derivatives with respect to x and y respectively for the function given by g(X,Y) = pX + qY – M i.e. the given constraint expressed as a function.
6. Find the value of the Determinant corresponding to the matrix HB i.e evaluate Det. HB

7. If **Det.HB > 0** then the critical values of X and Y obtained in step (4) corresponds to a **Maxima.**

   If **Det.HB < 0** then the critical values of X and Y obtained in step (4) corresponds to a **Minima.**

Illustration 8

Suppose a firm produces TV Sets at two different locations which produced x_1 and x_2 sets respectively. The joint cost function is given as $C = 0.1(x_1)^2 + 0.2(x_2)^2 + 0.2x_1x_2 + 180x_1 + 60x_2 + 25000$. If the firm has to supply 1000 sets of TV then find the number of sets to be produced in the two plants so that the joint cost is minimum.

Solution:

The firm has to supply 1000 sets in total. So $x_1 + x_2 = 1000$

Thus the above equation has become a constraint under which the plants are to work. So the problem becomes

Minimize $C = 0.1(x_1)^2 + 0.2(x_2)^2 + 0.2x_1x_2 + 180x_1 + 60x_2 + 25000$

Subject to $x_1 + x_2 = 1000$ Or, $x_1 + x_2 - 1000 = 0$ Or, $g(x_1, x_2) = 0$ [Let]

Using Lagrange's Multiplier (λ) we can get the Lagragian Function $L(x_1, x_2, \lambda) = C - \lambda.g(x_1, x_2)$

Substituting $C = 0.1(x_1)^2 + 0.2(x_2)^2 + 0.2x_1x_2 + 180x_1 + 60x_2 + 25000$ and $g(x_1, x_2) = x_1 + x_2 - 1000$ we rewrite the Lagrangian Function as $L(x_1, x_2, \lambda) = 0.1(x_1)^2 + 0.2(x_2)^2 + 0.2x_1x_2 + 180x_1 + 60x_2 + 25000 - \lambda(x_1 + x_2 - 1000)$

Differentiating the function partially with respect to x_1 we get, $0.2x_1 + 0.2x_2 + 180 - \lambda = L_{x_1}$

Differentiating the function partially with respect to x_2 we get, $0.4x_2 + 0.2x_1 + 60 - \lambda = L_{x_2}$

Differentiating the function partially with respect to λ we get, $-(x_1 + x_2 - 1000) = = L_x$

As per the 1st Order conditions of Lagrangian Method we have

$L_{x_1} = 0$ Or, $0.2x_1 + 0.2x_2 + 180 - \lambda = 0$ ---------- (1)

$L_{x_2} = 0$ Or, $0.4x_2 + 0.2x_1 + 60 - \lambda = 0$ ----------- (2)

$L_x = 0$ Or, $-(x_1 + x_2 - 1000) = 0$ Or, $x_1 + x_2 = 1000$ ------- (3)

From (1) we have, $0.2(x_1 + x_2) + 180 - \lambda = 0$ Or, $0.2(1000) + 180 - \lambda = 0$ [Since from (3), $x_1 + x_2 = 1000$]

Or, $\lambda = 380$ ------ (4)

From (2) we have, $0.2x_2 + 0.2(x_1 + x_2) + 60 - 380 = 0$ [Putting the value of λ from (4)]

Or, $0.2x_2 + 0.2(1000) - 320 = 0$ [Since from (3), $x_1 + x_2 = 1000$]

Or, $0.2x_2 = 120$ Or, $x_2 = 600$ sets

Putting $x_2 = 600$ in (3) we get $x_1 = 400$ sets

We have $g(x_1, x_2) = x_1 + x_2 - 1000$

Differentiating partially with respect to x_1 as well as x_2 we get $g_{x_1} = 1$ and $g_{x_2} = 1$

Again $L_{x_1} = 0.2x_1 + 0.2x_2 + 180 - \lambda$ gives $L_{x_1 x_2} = 0.2$ by partially differentiating with respect to x_1

Also $L_{x_1} = 0.2x_1 + 0.2x_2 + 180 - \lambda$ gives $L_{x_1 x_2} = 0.2$ by partially differentiating with respect to x_2

and $L_{x_2} = 0.4x_2 + 0.2x_1 + 60 - \lambda$ gives $L_{x_2 x_2} = 0.4$ by partially differentiating with respect to x_2

So the Bordered Hessian Matrix (HB) is given as $HB = \begin{bmatrix} 0 & g_{x_1} & g_{x_2} \\ g_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ g_{y_2} & L_{x_1x_2} & L_{x_2x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0.2 & 0.2 \\ 1 & 0.2 & 0.4 \end{bmatrix}$

Value of the determinant corresponding to Matrix HB is obtained by expanding with respect the first row as below

$$\text{Det. HB} = 0.\begin{vmatrix} 0.2 & 0.2 \\ 0.2 & 0.4 \end{vmatrix} - 1.\begin{vmatrix} 1 & 0.2 \\ 1 & 0.4 \end{vmatrix} + 1.\begin{vmatrix} 1 & 0.2 \\ 1 & 0.4 \end{vmatrix} = 0 - 1.(1x0.4 - 0.2x1) + 1.(1x0.2 - 0.2x1) = -0.2 < 0$$

As Det. HB < 0, there exist a minima corresponding to the critical values of $x_1 = 400$ sets & $x_2 = 600$ sets

Hence Cost minimisation can be possible by producing 400 TV Sets in 1st Plant & 600 Sets in 2nd Plant.

EXERCISE

A. Theoretical Questions:

Multiple Choice Questions

1. Optimization is the method of finding
   (a) The maximum point
   (b) The minimum point
   (c) The critical point
   (d) All of the above
2. Choose the correct answer
   (a) Optimization problems should have only one objective function
   (b) Constraint functions are compulsory for any optimization problem.
   (c) Objective function must be a continuous function
   (d) None of the above
3. The process of finding relative maximum or minimum of a function is known as
   (a) Optimization
   (b) Maximization
   (c) Minimization
   (d) Any of these
4. For a Cost Function $TC = 3Q^2 + 7Q + 12$, MC is –
   (a) 6Q
   (b) $6Q + 7$
   (c) $3Q + 12$
   (d) None of the above
5. MR is
   (a) First order derivative of TC
   (b) Second order derivative of TR
   (c) First order derivative of TR
   (d) Second order derivative of TC
6. In unconstrained optimization with single variable the sufficient condition for maximization is –
   (a) Second order derivative of the objective function must be zero.
   (b) Second order derivative of the objective function must be less than zero
   (c) Second order derivative of the objective function must be less than zero.
   (d) None of the above

7. In case of unconstrained optimization involving two variables the necessary condition is –

   (a) First order derivative of the objective function with respect to the variables should be zero.

   (b) First order partial derivative of the objective function with respect to the variables should be zero.

   (c) Either one of (a) and (b)

   (d) Both (a) and (b)

8. A Firm is said to achieve Condition of equilibrium when

   (a) Its objective is optimized.

   (b) Its profit is maximized.

   (c) Its loss is minimized

   (d) All of the above.

9. In the expression $D = AC – B^2$ used for describing the sufficient conditions for unconstrained optimization involving two variables (x and t), the meaning of A and C are –

   (a) 2nd order partial derivative of the objective function (f) with respect to x and y respectively.

   (b) 2nd order partial derivative of $\partial f/\partial x$ with respect to y

   (c) Both (a) and (b)

   (d) Only (a) but not (b)

10. A price discriminating Monopolist Firm operates in –

    (a) Such a Market where it is the sole supplier.

    (b) More than one Market.

    (c) Markets where it sells same product but in different prices.

    (d) All of the above.

11. In the expression $D = AC – B^2$ used for describing the sufficient conditions for a dual variable unconstrained optimization the term D is known as –

    (a) Hessian Matrix of order 2

    (b) Determinant for Hessian Matrix of order 2.

    (c) Matrix of partial derivatives of order 2.

    (d) Determinant of the Matrix of partial derivatives

12. For a dual plant Monopolist Firm with respective production costs C_1 & C_2 in the two plants, the necessary condition of equilibrium is

    (a) $MC_1 = MC_2 \neq MR$

    (b) $MC = MR$

    (c) $MC_1 = MC_2 = MR$

    (d) $MC_1 = MR_1$ & $MC_2 = MR_2$

13. Use of Lagrange's Multiplier is seen while –

    (a) Solving a problem of unconstrained optimization with single variable.

    (b) Solving a problem of optimization with inequality constraints.

(c) Solving a problem of optimization with one equality constraint.

(d) Solving a problem of optimization having no constraint.

14. For a firm the total cost function is $C(x) = -0.5x^2 + 11x + 600$. Which of the following statement is incorrect?

    (a) Average variable cost function is $AVC(x) = -0.5x + 11$

    (b) Marginal cost function is $MC(x) = -x + 11$

    (c) Cost of producing 10 units is ₹ 710/-

    (d) Average cost function is $AC(x) = -0.5x + 11 + 600/x$

15. Which one of the following statement is not correct?

    (a) Average Revenue of a Firm is same as the price at which its product is sold.

    (b) Total Profit is the product of quantity sold and the difference of Average Revenue and Average Cost.

    (c) When Marginal Revenue is zero then Total Revenue is maximum

    (d) None of the above.

Answers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d	a	a	b	c	b	b	d	d	d	b	c	c	c	d

State True or False

1. Profit maximization is not the sole objective of a Firm.
2. Static Optimization & Dynamic Optimization are two types of Unconstrained Optimization.
3. For unconstrained optimization involving single variable the sufficient condition to get a Minima is $\frac{d^2y}{dx^2}$ **< 0.**
4. Bordered Hessian Matrix is a square matrix of order 3.
5. If for a firm the Cost function is $C = wl + rk$. To minimize Cost subject to the constraint $y = k^{\alpha}.l^{\beta}$, the Lagrangian Function is given by $L = C - \lambda(k^{\alpha}.l^{\beta} - y)$
6. The critical values of x and y for optimization of the function $z = x^2 + y^2 + 0.5xy$ subject to $y = 90 - 2x$ are respectively $x = 39.375$ and $y = 11.25$
7. For multivariate unconstrained optimization involving two variables x and y, the necessary conditions require $\frac{df}{dx}$ as well as $\frac{df}{dy}$ are not equal.
8. The value of Hessian Determinant to be zero is the condition of getting a Saddle Point in the unconstrained optimization involving variables x and y.
9. Lagrangian Method can be used for constrained optimization problems involving two equality constraints.
10. The problem, Minimize $Z = -7x^2 + 6xy - 9y^2$ subject to $2x + y = 165$ can be solved only by using Lagragian Method.
11. If the Lagrangian function is $L = 4x^2 - 5xy + 6y^2 + \lambda(x + y - 30)$ then the critical values of x and y are 17 & 13 respectively.

12. Problems of constrained optimization with multiple equality constraints cannot be solved by Lagrangian Method.
13. A multiproduct firm has profit function given by $\pi = 100x - x^2 - 2xy + 200y - 3y^2$. The value of Hessian determinant will be(- 8).
14. D = 0 gives inconclusive result for a problem of dual variable unconstrained optimization, where D is the Hessian Determinant.
15. Price discriminating Monopolists operate in different markets with different products.

Answers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T	F	F	T	T	T	F	F	T	F	T	F	F	T	F

Fill in the blanks

1. Business is an activity or enterprise entered into for _____
2. Unconstrained and _____ optimization are the two situations under which optimization is carried out.
3. Unconstrained optimization is also known as ____ Maxima technique.
4. Calculus approach of ______ is used to optimize Objective Functions not subjected to any restriction.
5. $\frac{d^2y}{dx^2} = \mathbf{0}$ indicates existence of a _____ for the curve of the Objective Function.
6. $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ is known as _____ determinant and used in unconstrained optimization of multivariate function.
7. To optimize an objective function with single equality constraint _____ method is used.

Answers:

1.	Profit	**2.**	Constrained
3.	Unbounded	**4.**	Derivatives
5.	Point of Inflexion	**6.**	Hessian
7.	Lagrangian		

Short essay type questions

1. Explain the term – Multi-plant Monopolist
2. Using only the 1st Derivatives prove that MC = MR for any firm in Equilibrium.
3. If the Lagrangian of an optimization problem is $L = 8x^2 - 70x - 4xy - 50y + 5y^2 + \lambda(x + y - 35)$ then what is the original problem?

Essay type questions

1. Describe the steps involved in optimization of an Objective Function subject to an equality constraint.
2. Write down the necessary and sufficient conditions of unconstrained optimization of an Objective Function having (a) Single Variable and (b) Dual Variable.
3. Write down the necessary and sufficient conditions of Equilibrium of the following types of Monopolist Firms producing only one commodity – (i) Multi-plant and (ii) Price discriminating

B. Numerical Questions

Comprehensive Numerical Problems

1. A manufacturing company selling two wheelers (x) and three wheelers (y) has the following Demand functions $p_x = 40 – 0.02x – 0.01y$ and $p_y = 80 – 0.06y – 0.01x$. Find the Revenue maximizing levels of output and price of the Two Wheelers as well as Three Wheelers. What is the maximum Total Revenue? The prices are in ₹ '000. Ensure the Total Revenue obtained is maximum by 2nd Order derivative test.
2. The Production function of a Firm is given as $Q = f(K,L) = K^{0.5}L^{0.25}$ and the prices of Capital (K) and Labour (L) are respectively w and r. Derive Cost function of the Firm. Find the cost minimizing combination of Capital and Labour.
3. A manufacturer estimates his Annual Sales (in units) as a function of the expenditure made for Social Media and TV Advertising as $Z = 50000x + 40000y – 10x^2 – 20y^2 – 10xy$, where Z denotes Number of TVs sold per year and x & y denote Amounts spent on TV & Social Media Advertising in ₹ '000. Determine how much amounts should be spent on the two types of Advertising in order to maximize the number of TVs sold.
4. A monopolist offers 2 products which have the demand functions given as $q_1 = 14 – p_1/4$ & $q_2 = 24 – p_2/2$. The monopolist's joint cost function is $C(q_1, q_2) = (q_1)^2 + 5q_1q_2 + (q_2)^2$.Determine the quantities to be sold in order to maximize the Total Profit.
5. If the relation between Total Cost (y) and Output (x) is $y = 3x\left(\frac{x + 7}{x + 5}\right) + 5$, prove that the Marginal Cost falls continuously as the Output increases.

Answers:

1. Revenue maximising output of two and three wheelers are respectively 727 and 545 numbers and the corresponding prices are ₹ 20000/- and ₹ 40000/- Maximum Total Revenue = ₹ 36,340,000/-
2. Cost function is – C = rk + wl & Cost minimizing combination of Labour and Capital is given as
   $K = \sqrt[3]{[(2w/r)Q^4]}$ and $L = \sqrt[3]{[(r^2 / 4w^2)Q^4]}$
3. Amounts to be spent on TV and Social Media Advertising are respectively ₹ 2285720/- and ₹ 428570/-
4. The quantities to be sold are $q_1 = 2.75$ and $q_2 = 5.7$.

References:

- Trivedi M. L. – Managerial Economics – Theory and Applications, Tata Mcgraw – Hill Publishing Co. Ltd.
- Koutsoyiannis A – Modern Microeconomics (International edition, 2012), Macmillan Press Ltd.